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LIMIT SHADOWING WITH C^0 TRANSVERSALITY CONDITION

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Abstract. Let f be an Axiom A difeomorphism of a closed 2-dimensional smooth manifold M. We show that f has the limit shadowing property if and only if f satisfies the C^0 transversality condition.

1. Introduction

The notion of the pseudo-orbits very often appears in several branches of the modern theory of dynamical systems. For instance, the pseudo-orbit tracing property (shadowing property) usually plays an important role in stability theory (see [2]).

In this paper, we introduce the notion of the various shadowing property and basic definitions in the next section.

Pilyugin and Sakai [4] showed that if a diffeomorphism f of a closed smooth surface is Axiom A then the following statements are equivalents:

(a) f satisfies the C^0 transversality condition,
(b) f has the shadowing property, and
(c) f has the inverse shadowing property.

The present work aims to extend and complete some investigations of [4]. The main result of this paper is:

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Theorem A. Let M be a 2 dimensional smooth manifold, and let $f : M \to M$ be an Axiom A diffeomorphism. Then the following statements are equivalents:

(a) f satisfies the C^0 transversality condition,
(b) f has the shadowing property,
(c) f has the inverse shadowing property,
(d) f has the limit shadowing property, and
(e) f has the strong limit shadowing property.

The paper is organized as follows: in the next section we provide definitions and introduce notations for the paper. Especially, we can see that usual shadowing property is not limit shadowing property and the converse is not true. Moreover, our limit shadowing property([1]) is slightly different with the original limit shadowing property in [2]. In Section 3 we prove Theorem A.

2. Preliminaries

Let us be more precise. Let M be a closed 2-dimensional smooth manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ is called a *δ-pseudo-orbit* of $f \in \mathrm{Diff}(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is $y \in M$ such that $d(f^n(y), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. We introduce the limit shadowing property which is founded in [1]. We say that f has the *limit shadowing property* if there exists a $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is δ-limit pseudo orbit of f for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as $i \to -\infty$ hold, then there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Note that the limit shadowing property does not imply the shadowing property. In fact, in [1], this concept is called weak limit shadowing property (see, [1] Example 3, 4). It is different form the notion of Pilyugin [2]. Let S and $T \subset M$ be two C^1-smooth curves. Consider a point $x \in S \cap T$ and let C_η be the open disk of a small radius η in M centered at x. If $\eta > 0$ is sufficiently small, then S divides C_η into two open components homeomorphic to a disk; we denote these components by C_η^+ and C_η^-. We say that S and T are C^0 *transverse at x* if for any $\eta > 0$ the connected component of the intersection $T \cap C_\eta$ containing x intersects both disks C_η^+ and C_η^-. Note that C^0 transverse curves are not just homeomorphic images of transverse curves. For instance, consider the plane $\mathbb{R}^2$ with coordinates (x, y). Let S be the line $y = 0$, and let T be the graph of the function $f(x) = \exp(-1/x^2)\sin(1/x)$. Then S and T are C^0 transverse at $(0, 0)$, while $h(S)$ and $h(T)$ are not transverse at $h(0, 0)$ for any homeomorphism h. We denote by $\Omega(f)$ the set of nonwandering points, and $P(f)$ the set of periodic points. A closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

A map $f \in \mathrm{Diff}(M)$ satisfies *Axiom A* if $\Omega(f) = \overline{P(f)}$ is hyperbolic. Then there is a decomposition of $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$, where each Λ_i is the basic sets.

3. Proof of Theorem A

Let M be as before, and let f be an Axiom A diffeomorphism of M. As usual, we denote by $W^s(p)$ and $W^u(p)$ the stable and unstable mainfolds of a point $p \in \Omega(f)$ and by $W^s(\Lambda_i)$ and $W^u(\Lambda_i)$ the stable and unstable manifolds of a basic set Λ_i. We say that f satisfies the C^0 *transversality condition* if for any $p, q \in \Omega(f)$ with $\dim W^s(p) = \dim W^u(q) = 1$, the manifolds $W^s(p)$ and $W^u(q)$ are C^0 transverse at any point of their intersection.

Lemma 3.1. [4, Lemma 2.1] *Let $f \in \mathrm{Diff}(M)$ be an Axiom A. If f satisfies the C^0 transversality condition, then f is Ω-stable.*

From the above Lemma and [3, Lemma 1], we show that $(a) \Rightarrow (d)$ and $(a) \Rightarrow (e)$.

To show that $(d) \Rightarrow (a)$ and $(e) \Rightarrow (a)$, we need the following Lemmas and to prove these, we show that for any points $p, q \in \Omega(f)$,

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset,$$

where p and q are points of different basic sets. The following Lemma shows $(d) \Rightarrow (a)$.

Lemma 3.2. *Let $f \in \mathrm{Diff}(M)$ be an Axiom A. If f has the limit shadowing property then f satisfies the C^0 transversality condition.*

Proof. Let Λ_1 and Λ_2 be basic sets of f, and let $p \in \Lambda_1$ and $q \in \Lambda_2$. We will derive a contradiction. Suppose that a point $x \in W^s(p) \cap W^u(q)$ is not C^0 transverse. To simplify, we may assume that $f(p) = p$ and $f(q) = q$. Let $\delta > 0$ be the number of the limit shadowing property for f. Then we construct a δ-limit pseudo orbit as follows: Take a point $y \in W^u(p)$ such that $d(y, p) < \delta/2$. Then there are $l_1 > 0$, and $l_2 > 0$ such that $d(f^{-l_1}(x), q) < \delta$, and $d(f^{l_2}(x), p) < \delta/2$. Then

$$\xi = \{\ldots, q, q, f^{-l_1}(x), f^{-l_1+1}(x), \ldots, f^{-1}(x), x, f(x), \\ \ldots, f^{l_2-1}(x), y, f(y), \ldots\}$$

is a δ-limit pseudo orbit of f. Since f has the limit shadowing property, we can find a point $z \in M$ such that $d(f^i(z), x_i) \to 0$ as $i \to \pm\infty$. Here $x_i \in \xi$. Since $x \in W^s(p) \cap W^u(q)$ is not C^0 transverse, for any $\eta > 0$, we denote by $C^s_\eta(x) = B_\eta(x) \cap W^s(p)$ and $C^u_\eta(x) = B_\eta(x) \cap W^u(q)$. Let $B^+_\eta(x)$ and $B^-_\eta(x)$ be the components of $B_\eta(x) \setminus C^s_\eta(x)$. Then we may assume that $B^+_\eta(x) \cap C^u_\eta(x) = \emptyset$ and $B^-_\eta(x) \cap C^u_\eta(x) \neq \emptyset$.

If the point z belongs to the set $M \setminus B^-_\eta(x)$, then since $B^+_\eta(x) \cap C^u_\eta(x) = \emptyset$, $f^i(z) \not\to q$ as $i \to -\infty$. This is a contradiction. If the point z belongs to the set $M \setminus B^+_\eta(x)$ then since $B^-_\eta(x) \cap C^u_\eta(x) \neq \emptyset$, we consider two cases (i) $z \notin W^u(q)$, or (ii) $z \in W^u(q)$. The case (i), we can easily get a contradiction as in the above proof. In the case (ii), $f^i(z) \to q$ as $i \to -\infty$. But $f^i(z) \not\to y$ as $i \to +\infty$. This is a contradiction. Thus if f has the limit shadowing property then f satisfies C^0 transversality condition. □

Now, we introduce a notion of the strong limit shadowingproperty. Let Λ be a closed set. We say that f has the *strong limit shadowing property* on Λ if there exists a $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ is δ-limit pseudo orbit of f for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as $i \to -\infty$ hold, then there is a point $y \in \Lambda$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$.

The following Lemma proves that $(e) \Rightarrow (a)$.

LEMMA 3.3. *Let $f \in \mathrm{Diff}(M)$ be an Axiom A. If f has the strong limit shadowing property then f satisfies the C^0 transversality condition.*

Proof. The proof is just as same as the proof of Lemma 3.2. Indeed, first we can construct a pseudo orbit as in the above lemma, that is,

$$\xi = \{\ldots, q, q, f^{-l_1}(x), f^{-l_1+1}(x), \ldots, f^{-1}(x), x, f(x), \\ \ldots, f^{l_2-1}(x), y, f(y), \ldots, \}$$

is a δ-limit pseudo orbit of f. Since f has the strong limit shadowing property, every shadowing point of ξ must belong to the set Λ_1 or Λ_2 since $\Omega(f) = \{\Lambda_1, \Lambda_2\}$. Since Λ_1 and Λ_2 are basic sets, we know that for any $a \in \Lambda_i$, $f^i(a) \in \Lambda_i$. Thus if z is a strong limit shadowing point then the orbit of z stays in Λ_i for $i = 1, 2$. Therefore, f does not have the strong limit shadowing property. This is a contradiction. Thus if f has the strong limit shadowing property then f satisfies C^0 transversality condition. $\square$

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